8.6 Comparing Linear, Exponential, and Quadratic Functions

Essential Question

How can you compare the growth rates of linear, exponential, and quadratic functions?

EXPLORATION 1 Comparing Speeds

Work with a partner. Three cars start traveling at the same time. The distance traveled in t minutes is y miles. Complete each table and sketch all three graphs in the same coordinate plane. Compare the speeds of the three cars. Which car has a constant speed? Which car is accelerating the most? Explain your reasoning.

t	$y = t$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = 2^t - 1$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

t	$y = t^2$
0	
0.2	
0.4	
0.6	
0.8	
1.0	

COMPARING PREDICTIONS

To be proficient in math, you need to visualize the results of varying assumptions, explore consequences, and compare predictions with data.

EXPLORATION 2 Comparing Speeds

Work with a partner. Analyze the speeds of the three cars over the given time periods. The distance traveled in t minutes is y miles. Which car eventually overtakes the others?

t	$y = t$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

t	$y = 2^t - 1$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

t	$y = t^2$
1.0	
1.5	
2.0	
2.5	
3.0	
3.5	
4.0	
4.5	
5.0	

Communicate Your Answer

3. How can you compare the growth rates of linear, exponential, and quadratic functions?

4. Which function has a growth rate that is eventually much greater than the growth rates of the other two functions? Explain your reasoning.

8.6 Lesson

Core Vocabulary

average rate of change, *p. 462*

Previous

slope

What You Will Learn

- Choose functions to model data.
- Write functions to model data.
- Compare functions using average rates of change.
- Solve real-life problems involving different function types.

Choosing Functions to Model Data

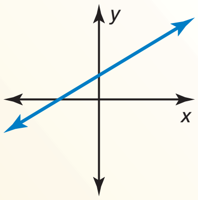
So far, you have studied linear functions, exponential functions, and quadratic functions. You can use these functions to model data.

Core Concept

Linear, Exponential, and Quadratic Functions

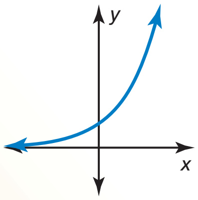
Linear Function

$y = mx + b$



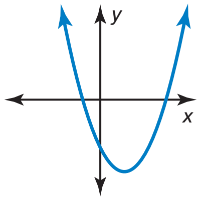
Exponential Function

$y = ab^x$



Quadratic Function

$y = ax^2 + bx + c$



EXAMPLE 1 Using Graphs to Identify Functions

Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

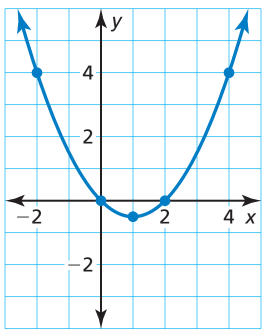
a. $(4, 4), (2, 0), (0, 0), \left(1, -\frac{1}{2}\right), (-2, 4)$

b. $(0, 1), (2, 4), (4, 7), (-2, -2), (-4, -5)$

c. $(0, 2), (2, 8), (1, 4), (-1, 1), \left(-2, \frac{1}{2}\right)$

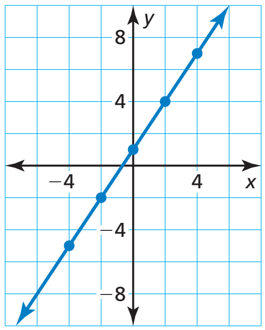
SOLUTION

a.



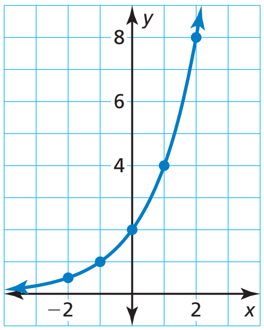
▶ quadratic

b.



▶ linear

c.



▶ exponential

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Plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

1. $(-1, 5), (2, -1), (0, -1), (3, 5), (1, -3)$

2. $(-1, 2), (-2, 8), (-3, 32), \left(0, \frac{1}{2}\right), \left(1, \frac{1}{8}\right)$

3. $(-3, 5), (0, -1), (2, -5), (-4, 7), (1, -3)$

Core Concept

Differences and Ratios of Functions

You can use patterns between consecutive data pairs to determine which type of function models the data. The differences of consecutive y-values are called *first differences*. The differences of consecutive first differences are called *second differences*.

- **Linear Function** The first differences are constant.
- **Exponential Function** Consecutive y-values have a common *ratio*.
- **Quadratic Function** The second differences are constant.

In all cases, the differences of consecutive x-values need to be constant.

STUDY TIP

The first differences for exponential and quadratic functions are *not* constant.

EXAMPLE 2 Using Differences or Ratios to Identify Functions

Tell whether each table of values represents a *linear*, an *exponential*, or a *quadratic* function.

a.

x	−3	−2	−1	0	1
y	11	8	5	2	−1

b.

x	−2	−1	0	1	2
y	1	2	4	8	16

c.

x	−2	−1	0	1	2
y	−1	−2	−1	2	7

STUDY TIP

First determine that the differences of consecutive x-values are constant. Then check whether the first differences are constant or consecutive y-values have a common ratio. If neither of these is true, check whether the second differences are constant.

SOLUTION

a.

+1 +1 +1 +1

x	−3	−2	−1	0	1
y	11	8	5	2	−1

−3 −3 −3 −3

▶ The first differences are constant. So, the table represents a linear function.

b.

+1 +1 +1 +1

x	−2	−1	0	1	2
y	1	2	4	8	16

×2 ×2 ×2 ×2

▶ Consecutive y-values have a common ratio. So, the table represents an exponential function.

c.

+1 +1 +1 +1

x	−2	−1	0	1	2
y	−1	−2	−1	2	7

first differences → −1 +1 +3 +5

second differences → +2 +2 +2

▶ The second differences are constant. So, the table represents a quadratic function.

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4. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function.

x	−1	0	1	2	3
y	1	3	9	27	81

Writing Functions to Model Data

EXAMPLE 3 **Writing a Function to Model Data**

x	2	4	6	8	10
y	12	0	−4	0	12

Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

SOLUTION

Step 1 Determine which type of function the table of values represents.

x	2	4	6	8	10
y	12	0	−4	0	12

+2 +2 +2 +2

first differences → −12 −4 +4 +12

second differences → +8 +8 +8

The second differences are constant. So, the table represents a quadratic function.

Step 2 Write an equation of the quadratic function. Using the table, notice that the *x*-intercepts are 4 and 8. So, use intercept form to write a function.

$y = a(x - 4)(x - 8)$ Substitute for *p* and *q* in intercept form.

Use another point from the table, such as (2, 12), to find *a*.

$12 = a(2 - 4)(2 - 8)$ Substitute 2 for *x* and 12 for *y*.

$1 = a$ Solve for *a*.

Use the value of *a* to write the function.

$y = (x - 4)(x - 8)$ Substitute 1 for *a*.

$= x^2 - 12x + 32$ Use the FOIL Method and combine like terms.

STUDY TIP

To check your function in Example 3, substitute the other points from the table to verify that they satisfy the function.

So, the quadratic function is $y = x^2 - 12x + 32$.

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5. Tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. Then write the function.

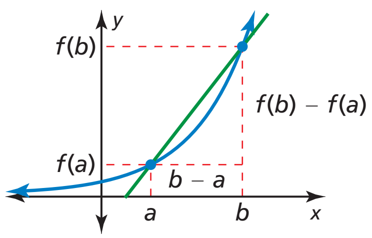
x	−1	0	1	2	3
y	16	8	4	2	1

Comparing Functions Using Average Rates of Change

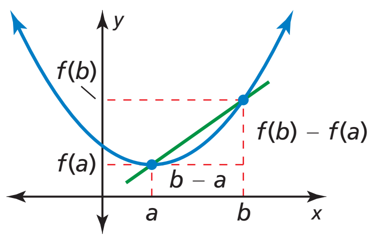
For nonlinear functions, the rate of change is not constant. You can compare two nonlinear functions over the same interval using their *average rates of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$.

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

Exponential Function



Quadratic Function



Core Concept

Comparing Functions Using Average Rates of Change

- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing exponential function $y = f(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ or an increasing linear function $y = h(x)$. So, as x increases, $f(x)$ will eventually exceed $g(x)$ or $h(x)$.
- As a and b increase, the average rate of change between $x = a$ and $x = b$ of an increasing quadratic function $y = g(x)$ will eventually exceed the average rate of change between $x = a$ and $x = b$ of an increasing linear function $y = h(x)$. So, as x increases, $g(x)$ will eventually exceed $h(x)$.

STUDY TIP

You can explore these concepts using a graphing calculator.

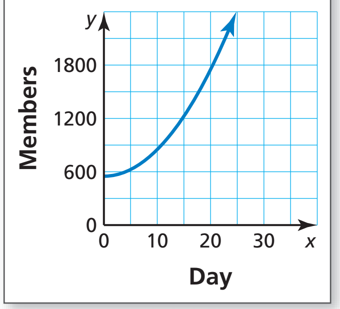
EXAMPLE 4 Using and Interpreting Average Rates of Change

Two social media websites open their memberships to the public. (a) Compare the websites by calculating and interpreting the average rates of change from Day 10 to Day 20. (b) Predict which website will have more members after 50 days. Explain.

Website A

Day, x	Members, y
0	650
5	1025
10	1400
15	1775
20	2150
25	2525

Website B



SOLUTION

a. Calculate the average rates of change by using the points whose x-coordinates are 10 and 20.

Website A: Use (10, 1400) and (20, 2150).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} = \frac{2150 - 1400}{20 - 10} = \frac{750}{10} = 75$$

Website B: Use the graph to estimate the points when $x = 10$ and $x = 20$. Use (10, 850) and (20, 1800).

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a} \approx \frac{1800 - 850}{20 - 10} = \frac{950}{10} = 95$$

From Day 10 to Day 20, Website A membership increases at an average rate of 75 people per day, and Website B membership increases at an average rate of about 95 people per day. So, Website B membership is growing faster.

b. Using the table, membership increases and the average rates of change are constant. So, Website A membership can be represented by an increasing linear function. Using the graph, membership increases and the average rates of change are increasing. It appears that Website B membership can be represented by an increasing exponential or quadratic function.

After 25 days, the memberships of both websites are about equal and the average rate of change of Website B exceeds the average rate of change of Website A. So, Website B will have more members after 50 days.

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6. Compare the websites in Example 4 by calculating and interpreting the average rates of change from Day 0 to Day 10.

Solving Real-Life Problems

EXAMPLE 5 Comparing Different Function Types

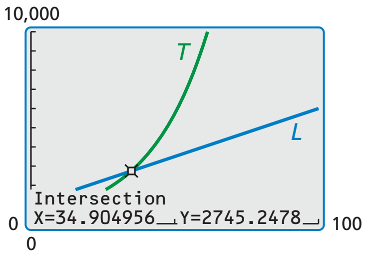
In 1900, Littleton had a population of 1000 people. Littleton's population increased by 50 people each year. In 1900, Tinyville had a population of 500 people. Tinyville's population increased by 5% each year.

a. In what year were the populations about equal?

b. Suppose Littleton's initial population doubled to 2000 and maintained a constant rate of increase of 50 people each year. Did Tinyville's population still catch up to Littleton's population? If so, in which year?

c. Suppose Littleton's rate of increase doubled to 100 people each year, in addition to doubling the initial population. Did Tinyville's population still catch up to Littleton's population? Explain.

SOLUTION



a. Let x represent the number of years since 1900. Write a function to model the population of each town.

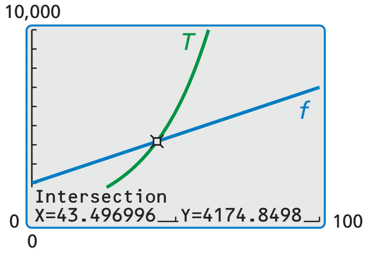
Littleton: $L(x) = 50x + 1000$ — Linear function

Tinyville: $T(x) = 500(1.05)^x$ — Exponential function

Use a graphing calculator to graph each function in the same viewing window. Use the *intersect* feature to find the value of x for which $L(x) \approx T(x)$. The graphs intersect when $x \approx 34.9$.

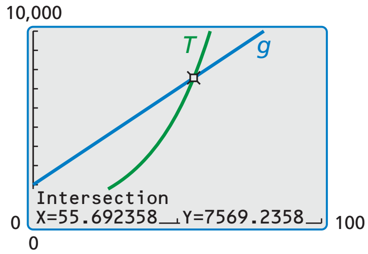
▶ So, the populations were about equal in 1934.

b. Littleton's new population function is $f(x) = 50x + 2000$. Use a graphing calculator to graph f and T in the same viewing window. Use the *intersect* feature to find the value of x for which $f(x) \approx T(x)$. The graphs intersect when $x \approx 43.5$.



▶ So, Tinyville's population caught Littleton's population in 1943.

c. Littleton's new population function is $g(x) = 100x + 2000$. Use a graphing calculator to graph g and T in the same viewing window. Use the *intersect* feature to find the value of x for which $g(x) \approx T(x)$. The graphs intersect when $x \approx 55.7$.



▶ So, Tinyville's population caught Littleton's population in 1955. Because Littleton's population shows linear growth and Tinyville's population shows exponential growth, Tinyville's population eventually exceeded Littleton's regardless of Littleton's constant rate or initial value.

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7. WHAT IF? Tinyville's population increased by 8% each year. In what year were the populations about equal?

8.6 Exercises

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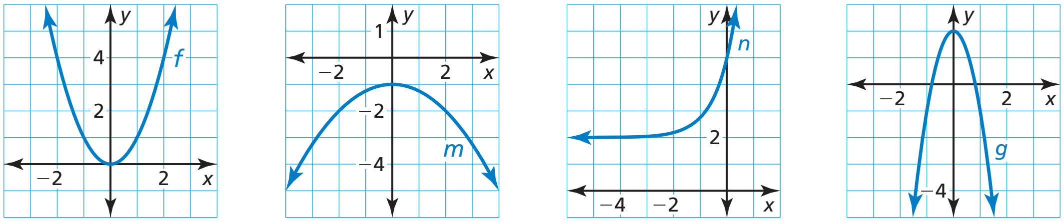
Vocabulary and Core Concept Check

1. **WRITING** Name three types of functions that you can use to model data. Describe the equation and graph of each type of function.

2. **WRITING** How can you decide whether to use a linear, an exponential, or a quadratic function to model a data set?

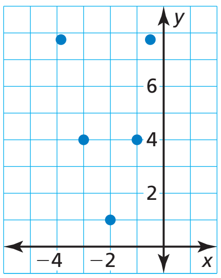
3. **VOCABULARY** Describe how to find the average rate of change of a function $y = f(x)$ between $x = a$ and $x = b$.

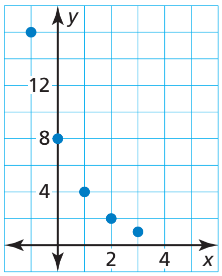
4. **WHICH ONE DOESN'T BELONG?** Which graph does *not* belong with the other three? Explain your reasoning.

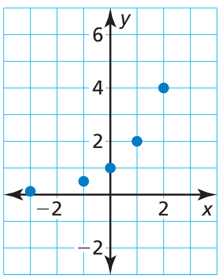


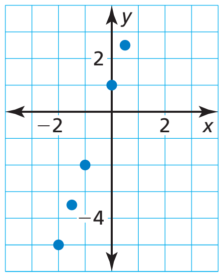
Monitoring Progress and Modeling with Mathematics

In Exercises 5–8, tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function.

5. 

6. 

7. 

8. 

In Exercises 9–14, plot the points. Tell whether the points appear to represent a *linear*, an *exponential*, or a *quadratic* function. *(See Example 1.)*

9. $(-2, -1), (-1, 0), (1, 2), (2, 3), (0, 1)$

10. $\left(0, \frac{1}{4}\right), (1, 1), (2, 4), (3, 16), \left(-1, \frac{1}{16}\right)$

11. $(0, -3), (1, 0), (2, 9), (-2, 9), (-1, 0)$

12. $(-1, -3), (-3, 5), (0, -1), (1, 5), (2, 15)$

13. $(-4, -4), (-2, -3.4), (0, -3), (2, -2.6), (4, -2)$

14. $(0, 8), (-4, 0.25), (-3, 0.4), (-2, 1), (-1, 3)$

In Exercises 15–18, tell whether the table of values represents a *linear*, an *exponential*, or a *quadratic* function. *(See Example 2.)*

15.

x	−2	−1	0	1	2
y	0	0.5	1	1.5	2

16.

x	−1	0	1	2	3
y	0.2	1	5	25	125

17.

x	2	3	4	5	6
y	2	6	18	54	162

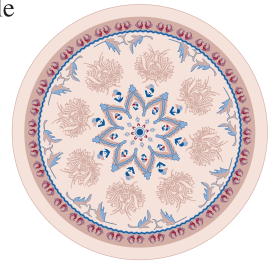
18.

x	−3	−2	−1	0	1
y	2	4.5	8	12.5	18

19. **MODELING WITH MATHEMATICS** A student takes a subway to a public library. The table shows the distances d (in miles) the student travels in t minutes. Let the time t represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.

Time, *t*	0.5	1	3	5
Distance, *d*	0.335	0.67	2.01	3.35

20. **MODELING WITH MATHEMATICS** A store sells custom circular rugs. The table shows the costs c (in dollars) of rugs that have diameters of d feet. Let the diameter d represent the independent variable. Tell whether the data can be modeled by a *linear*, an *exponential*, or a *quadratic* function. Explain.



Diameter, *d*	3	4	5	6
Cost, *c*	63.90	113.60	177.50	255.60

In Exercises 21–26, tell whether the data represent a *linear*, an *exponential*, or a *quadratic* function. Then write the function. *(See Example 3.)*

21. (−2, 8), (−1, 0), (0, −4), (1, −4), (2, 0), (3, 8)

22. (−3, 8), (−2, 4), (−1, 2), (0, 1), (1, 0.5)

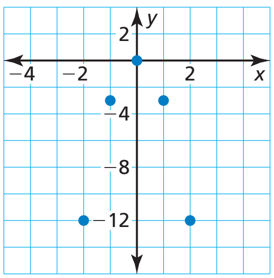
23.

x	−2	−1	0	1	2
y	4	1	−2	−5	−8

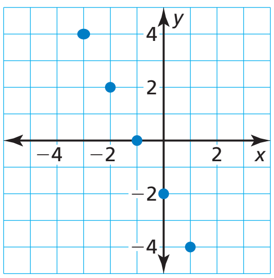
24.

x	−1	0	1	2	3
y	2.5	5	10	20	40

25.



26.



27. **ERROR ANALYSIS** Describe and correct the error in determining whether the table represents a linear, an exponential, or a quadratic function.

✗

+1 +1 +1 +1

x	1	2	3	4	5
y	3	9	27	81	243

×3 ×3 ×3 ×3

Consecutive y-values change by a constant amount. So, the table represents a linear function.

28. **ERROR ANALYSIS** Describe and correct the error in writing the function represented by the table.

✗

+1 +1 +1 +1

x	−3	−2	−1	0	1
y	4	0	−2	−2	0

first differences → −4 −2 +0 +2

second differences → +2 +2 +2

The table represents a quadratic function.

$$f(x) = a(x-2)(x-1)$$
$$4 = a(-3-2)(-3-1)$$
$$\frac{1}{5} = a$$
$$f(x) = \frac{1}{5}(x-2)(x-1)$$
$$= \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$$

So, the function is $f(x) = \frac{1}{5}x^2 - \frac{3}{5}x + \frac{2}{5}$.

29. **REASONING** The table shows the numbers of people attending the first five football games at a high school.

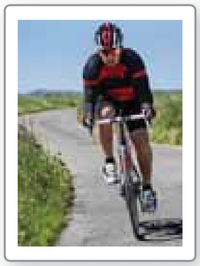
Game, *g*	1	2	3	4	5
People, *p*	252	325	270	249	310

a. Plot the points. Let the game g represent the independent variable.

b. Can a linear, an exponential, or a quadratic function represent this situation? Explain.

30. **MODELING WITH MATHEMATICS** The table shows the breathing rates y (in liters of air per minute) of a cyclist traveling at different speeds x (in miles per hour).

Speed, x	20	21	22	23	24
Breathing rate, y	51.4	57.1	63.3	70.3	78.0



a. Plot the points. Let the speed x represent the independent variable. Then determine the type of function that best represents this situation.

b. Write a function that models the data.

c. Find the breathing rate of a cyclist traveling 18 miles per hour. Round your answer to the nearest tenth.

31. **ANALYZING RATES OF CHANGE** The function $f(t) = -16t^2 + 48t + 3$ represents the height (in feet) of a volleyball t seconds after it is hit into the air.

a. Copy and complete the table.

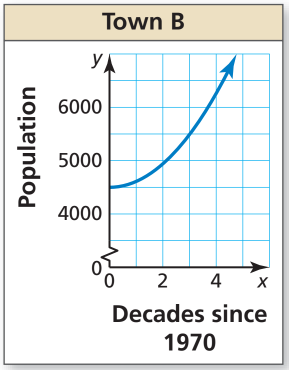
t	0	0.5	1	1.5	2	2.5	3
$f(t)$							

b. Plot the ordered pairs and draw a smooth curve through the points.

c. Describe where the function is increasing and decreasing.

d. Find the average rate of change for each 0.5-second interval in the table. What do you notice about the average rates of change when the function is increasing? decreasing?

32. **ANALYZING RELATIONSHIPS** The population of Town A in 1970 was 3000. The population of Town A increased by 20% every decade. Let x represent the number of decades since 1970. The graph shows the population of Town B. *(See Example 4.)*

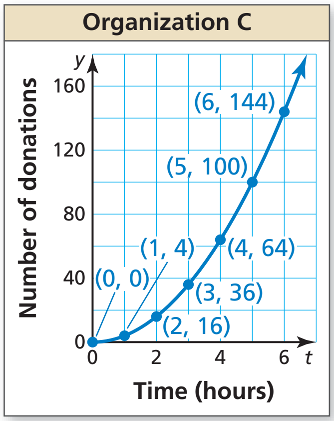


a. Compare the populations of the towns by calculating and interpreting the average rates of change from 1990 to 2010.

b. Predict which town will have a greater population after 2030. Explain.

33. **ANALYZING RELATIONSHIPS** Three organizations are collecting donations for a cause. Organization A begins with one donation, and the number of donations quadruples each hour. The table shows the numbers of donations collected by Organization B. The graph shows the numbers of donations collected by Organization C.

Time (hours), t	Number of donations, y
0	0
1	4
2	8
3	12
4	16
5	20
6	24



a. What type of function represents the numbers of donations collected by Organization A? B? C?

b. Find the average rates of change of each function for each 1-hour interval from $t = 0$ to $t = 6$.

c. For which function does the average rate of change increase most quickly? What does this tell you about the numbers of donations collected by the three organizations?

34. **COMPARING FUNCTIONS** The room expenses for two different resorts are shown. *(See Example 5.)*

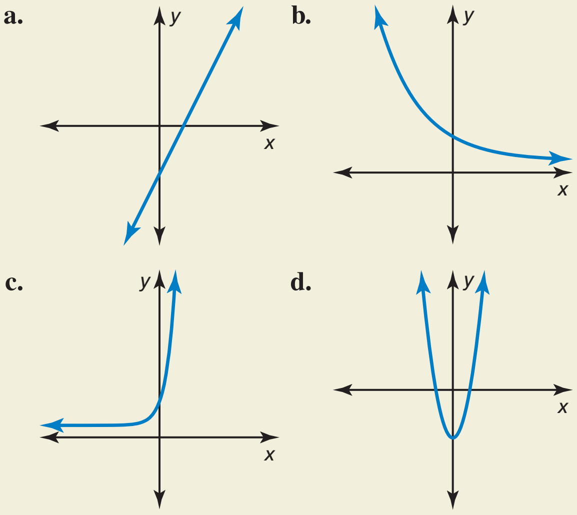
a. For what length of vacation does each resort cost about the same?

b. Suppose Blue Water Resort charges $1450 for the first three nights and $105 for each additional night. Would Sea Breeze Resort ever be more expensive than Blue Water Resort? Explain.

c. Suppose Sea Breeze Resort charges $1200 for the first three nights. The charge increases 10% for each additional night. Would Blue Water Resort ever be more expensive than Sea Breeze Resort? Explain.

35. REASONING Explain why the average rate of change of a linear function is constant and the average rate of change of a quadratic or exponential function is not constant.

36. HOW DO YOU SEE IT? Match each graph with its function. Explain your reasoning.



A. $y = 2x^2 - 4$

B. $y = 2(4)^x + 1$

C. $y = 2\left(\frac{3}{4}\right)^x + 1$

D. $y = 2x - 4$

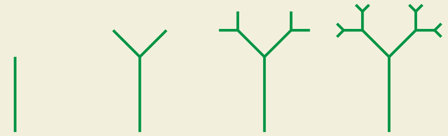
37. CRITICAL THINKING In the ordered pairs below, the y-values are given in terms of n. Tell whether the ordered pairs represent a *linear*, an *exponential*, or a *quadratic* function. Explain.

$(1, 3n - 1), (2, 10n + 2), (3, 26n),$
$(4, 51n - 7), (5, 85n - 19)$

38. USING STRUCTURE Write a function that has constant second differences of 3.

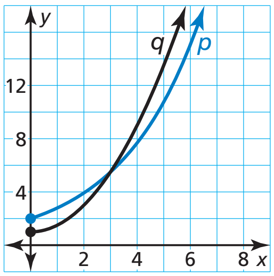
39. CRITICAL THINKING Is the graph of a set of points enough to determine whether the points represent a linear, an exponential, or a quadratic function? Justify your answer.

40. THOUGHT PROVOKING Find four different patterns in the figure. Determine whether each pattern represents a *linear*, an *exponential*, or a *quadratic* function. Write a model for each pattern.



$n = 1$ $n = 2$ $n = 3$ $n = 4$

41. MAKING AN ARGUMENT Function p is an exponential function and function q is a quadratic function. Your friend says that after about $x = 3$, function q will always have a greater y-value than function p. Is your friend correct? Explain.



42. USING TOOLS The table shows the amount a (in billions of dollars) United States residents spent on pets or pet-related products and services each year for a 5-year period. Let the year x represent the independent variable. Using technology, find a function that models the data. How did you choose the model? Predict how much residents will spend on pets or pet-related products and services in Year 7.

Year, x	1	2	3	4	5
Amount, a	53.1	56.9	61.8	65.7	67.1

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Evaluate the expression. *(Section 6.2)*

43. $\sqrt{121}$

44. $\sqrt[3]{125}$

45. $\sqrt[3]{512}$

46. $\sqrt[5]{243}$

Find the product. *(Section 7.3)*

47. $(x + 8)(x - 8)$

48. $(4y + 2)(4y - 2)$

49. $(3a - 5b)(3a + 5b)$

50. $(-2r + 6s)(-2r - 6s)$